3/12/20

Πχ

Άσκηση 5, σελ. 95: $y''' - 3y'' + 2y' = e^t \to$ Β.σ.λ. $\to y_\mu$, $y = z \cdot e^t$

Λύση: $y = z\cdot e^t \rightsquigarrow e^t \cdot z''' + 3e^t \cdot z'' + 3e^t z' + z \cdot e^t - 3(e^t z'' + 2e^t z' + e^t) + 2(e^t z' + e^t z) = e^t$.

$\Rightarrow z''' + 3z'' + 3z' + z - 3z'' - 6z' - 3z + 2z' + 2z = 1$

$\Rightarrow z''' - z' = 1 \rightsquigarrow z_\mu = at + b \;:\; -a = 1 \Rightarrow a = -1$.

$\Rightarrow z_\mu = -t \to y_\mu = -te^t$ (για $b=0$)

Λύσεις Ομογενούς: $\lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda^2 - 3\lambda + 2)$, $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$

Β.σ.λ. $= \{1, e^t, e^{2t}\}$

$\Rightarrow y(t) = c_1 + c_2 \cdot e^t + c_3 \cdot e^{2t} - t \cdot e^t$.

Πχ

Παράδειγμα 3νι, σελ. 110: $y'' - 4y' + 4y = \sin x \quad (*)$

Λύση: (E_0): $y'' - 4y' + 4y = 0$

$\Rightarrow P(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 \to \lambda = 2$ διπλή ρίζα.

Β.σ.λ. $= \{e^{2t}, t \cdot e^{2t}\}$

Η $(*)$ γίνεται: $y'' - 4y' + 4y = e^{0t}(\cos t + i \sin t)$ // Im y.

$y'' - 4y' + 4y = e^{it}\ \tilde{y} \rightsquigarrow \tilde{y} = z \cdot e^t$:

$e^{it} z'' + 2z' \cdot i \cdot e^{it} + z(-1)e^{it} - 4(z' e^{it} + iz \cdot e^{it}) + 4z e^{it} = e^{it}$

$\Rightarrow z'' + 2iz' - z - 4z' - 4iz + 4z = 1$

$\Rightarrow z'' + z'(2i - 4) + z(-1 - 4i + 4) = 1$

$\Rightarrow z(3 - 4i) = 1$

$\rightsquigarrow z_\mu = \dfrac{3+4i}{25} = \dfrac{1}{25}(3+4i)$, $\tilde{y}_\mu = \dfrac{1}{25}(3+4i)(\cos t + i \sin t)$

$\Rightarrow y_\mu = \operatorname{Im} \tilde{y}_\mu = \operatorname{Im} \dfrac{1}{25}(3+4i)(\cos t + i\sin t) = \dfrac{1}{25}(4\cos t + 3\sin t)$

(Πχ) **Παράδειγμα:** $y^{(4)}+2y''+y=\cos 3t$.

Λύση: $P(\lambda)=\lambda^4+2\lambda^2+1=(\lambda^2+1)\to \pm i$ διπλή ρίζα.

Β.σ.λ. $=\{\cos t, \sin t, t\cos t, t\sin t\}$

$\hookrightarrow y^{(4)}+2y''+y=e^{3ti} \to y=z\cdot e^{3ti}$

$\Rightarrow e^{3ti}z^{(4)}+4\cdot z'''\cdot 3i\cdot e^{3ti}+\binom{4}{2}z''\cdot 9(-1)e^{ti}+4z'\cdot 27(-i)e^{3ti}+z\cdot 81\cdot e^{3ti}$ ($\binom{4}{2}=6$)

$+2(z''e^{3ti}+2z'\cdot 3i\cdot e^{3ti}+z\cdot(-9)e^{3ti})+z\cdot e^{3ti}=e^{3ti}$

$\Rightarrow z^{(4)}+12iz'''-54z''-108iz'+81z+2z''+12iz'-18z+z=1$

$\to z_\mu=\frac{1}{64}$

$\Rightarrow \tilde{y}_\mu=\frac{1}{64}e^{3ti} \longrightarrow y_\mu=\mathrm{Re}\,\tilde{y}_\mu=\frac{1}{64}\cos 3t=y_\mu$.

(Πχ) **Άσκηση 4ii, σελ. 113:** $y'''+y'=t$, $y(0)=0$, $y'(0)=2$, $y''(0)=0$.

Λύση: Χαρακτηριστικό πολυώνυμο: $\lambda^3+\lambda=\lambda(\lambda^2+1)\to\lambda_1=0, \lambda_{2,3}=\pm i$

Β.σ.λ. $=\{1, \cos t, \sin t\}$

$y_\mu'=at+\beta \Rightarrow a=1, \beta=0 \Leftarrow y_\mu'=t \rightsquigarrow y_\mu=\frac{t^2}{2}$

$\to y(t)=\frac{t^2}{2}+c_1+c_2\cdot\cos t+c_3\sin t$.

(Πχ) **Άσκηση Β-40:** $a\cdot y''+by'+cy=e^{-kt}$, $a,b,c,k>0$, $bk\neq ak^2+c$.

$\rightsquigarrow$ όλες οι λύσεις $\to 0$

Λύση: Χ.Π.: $a\lambda^2+b\lambda+c \rightsquigarrow \lambda_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$.

• $\Delta>0$: $\lambda_1,\lambda_2<0 \rightsquigarrow e^{\lambda_1 t}, e^{\lambda_2 t}\to 0$ (με $t\to\infty$)

• $\Delta=0$: $\lambda=\frac{-b}{2a}<0 \rightsquigarrow e^{\lambda t}, te^{\lambda t}\to 0$

• $\Delta<0$: $e^{-b/2at}\cos(\dots t), \sin(\dots t)$, $e^{-b/2at}\downarrow 0$

Β.σ.λ. $=\{y_1, y_2\}\xrightarrow{t\to\infty}0$

$y = z \cdot e^{-kt}$ $\rightsquigarrow$ a.

$\rightsquigarrow a\left[z'' \cdot e^{-kt} + 2(-k)z' \cdot e^{-kt} + k^2 \cdot z \cdot e^{-kt}\right] + b(z'e^{-kt} - k \cdot z \cdot e^{-kt}) + c \cdot z \cdot e^{-kt} = e^{-kt}$

$\rightsquigarrow az'' + z'(-2ka + b) + z(ak^2 - kb + c) = 1$

$\Rightarrow z_\mu = \dfrac{1}{ak^2 - kb + c} \rightsquigarrow y_\mu = \dfrac{1}{ak^2 - bk + c} \quad e^{-kt} \to 0$

Ασκηση Β-28 : $y'' + 8y' + 25y = 2 \cdot \cos t$. Νδο $\exists a, \delta \in \mathbb{R}$:
$\lim [y(t) - a\cos(t - \delta)] = 0$, $\forall y$ λύση.

Λύση: Χ.Π. : $P(\lambda) = \lambda^2 + 8\lambda + 25 = 0$

$\rightsquigarrow \lambda_{1,2} = \dfrac{-8 \pm \sqrt{64 - 100}}{2} = \dfrac{-8 \pm 6i}{2} = -4 \pm 3i$

Β.σ.λ. $= \{e^{-4t}\cos 3t, e^{-4t}\sin 3t\}$

$\tilde{y}'' + 8\tilde{y}' + 25\tilde{y} = 2e^{it}$ $(Re\tilde{y})$

$\tilde{y} = z \cdot e^{it} \to 24z + 8iz = 2 \to z_\mu = \dfrac{1}{40}(3 - i)$

$y_\mu = Re\left\{\dfrac{1}{40}(3-i)e^{it}\right\} = Re\left\{\dfrac{1}{40}(3-i)(\cos t + i\sin t)\right\} = \dfrac{1}{40}(3\cos t + \sin t)$

Αρα $y(t) = c_1 \cdot e^{-4t} \cdot \cos 3t + c_2 \cdot e^{-4t} \cdot \sin 3t + \dfrac{1}{40}(3\cos t + \sin t)$.

$\underbrace{y(t) = \dfrac{1}{40}(3\cos t + \sin t)}_{\text{θα το γράψω στη μορφή } a\cdot\cos(t-\delta)} \overset{*}{=} c_1 \cdot e^{-4t} \cdot \cos 3t + c_2 \cdot e^{-4t} \cdot \sin 3t \to 0$

$\Rightarrow 3 \cdot \cos t + \sin t = \sqrt{10}\left(\dfrac{3}{\sqrt{10}}\cos t + \dfrac{1}{\sqrt{10}}\sin t\right) = \sqrt{10}(\cos\delta \cdot \cos t + \sin\delta \cdot \sin t) =$
$= \sqrt{10} \cdot \cos(t - \delta)$.

$\left[\delta: \cos\delta = \dfrac{3}{\sqrt{10}}, \sin\delta = \dfrac{1}{\sqrt{10}}\right]$

$*$: $\dfrac{\sqrt{10}}{40}\cos(t - \delta)$

$a\cos t + b\sin t = \sqrt{a^2 + b^2}\left(\dfrac{a}{\sqrt{a^2+b^2}}\cos t + \dfrac{b}{\sqrt{a^2+b^2}}\sin t\right)$

Άσκηση B-47: $y'' + 2ay' + by = \varphi(t)$, $a>0, a^2<b$, φ: φραγμένη.

Λύση: Λύσεις φραγμένες με φραγμένη παράγωγο.

Χ.Π.: $\lambda^2 + 2a\lambda + b \rightarrow \lambda_{1,2} = \frac{-2a \pm \sqrt{\overset{<0}{4a^2-4b}}}{2} = -a \pm \sqrt{a^2-b} = -a \pm i\underset{\sqrt{b-a^2}}{\underset{\parallel}{\delta}}$

$B.\sigma.\lambda. = \{e^{-at} \cdot \cos\delta t, \ e^{-at} \cdot \sin\delta t\}$.

$$W(x) = \begin{vmatrix} e^{-at}\cos\delta t & e^{-at}\sin\delta t \\ -a \cdot e^{-at}\cos\delta t - \delta \cdot e^{-at}\sin\delta t & -a \cdot e^{-at} \cdot \sin\delta t + e^{-at} \cdot \delta \cdot \cos\delta t \end{vmatrix} =$$

$$= e^{-at}(\delta \cdot \cos^2\delta t - a \cdot \sin t \cos\delta t + a \cdot \sin\delta \cdot \cos\delta t + \delta \cdot \sin^2\delta t) = e^{-at} \cdot \delta$$

$$W_1(x) = \begin{vmatrix} 0 & e^{-at}\sin\delta t \\ 1 & 0 \end{vmatrix} = -e^{-at}\sin\delta t$$

$$W_2(x) = \begin{vmatrix} e^{-at}\cos\delta t & 0 \\ 0 & 1 \end{vmatrix} = e^{-at} \cdot \cos\delta t$$

$$y_\mu(t) = e^{-at} \cdot \cos(\delta t) \int_0^t \frac{-e^{-as}\sin(\delta s)}{\delta \cdot e^{-2as}} \cdot \frac{\varphi(s)}{1} ds + \quad (*)$$

$$+ e^{-at} \cdot \sin(\delta t) \cdot \int_0^t \frac{e^{-as}\cos(\delta s)}{\delta \cdot e^{-2as}} \cdot \frac{\varphi(s)}{1} ds$$

$(*)$ $$e^{-at} \cdot 1 \int_0^t \frac{1}{\delta} e^{as} \|\varphi(s)\| ds \leq e^{-at} \cdot \frac{M}{\delta} \int_0^t e^{as} ds =$$

$$= \frac{M}{\delta} \cdot e^{-at}\left[\frac{e^{at}-1}{a}\right] \leq \frac{M}{\delta a}$$

Αντίστοιχα, για το 2ο κομμάτι.